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Finite strain in simple shear, inspected with Mohr circles for stretch

REINOUD L. M. VISSERS

Department of Geology, IVA, Budapestlaan 4, Utrecht, The Netherlands

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Abstract—The orientation and magnitude of the finite strain, built up in simple shear, and the orientation and rotation of material lines eventually parallel to the long principal axis of the strain ellipse have simple relationships with the shear strain. Off-axis Mohr circles for stretch and reciprocal stretch allow convenient proof of such relationships, whilst some of these are not often seen in existing structural literature. In particular, it is noted that the shear strain in ideal simple shear equals the difference of the principal stretches.

INTRODUCTION

IN studies concerned with geometrical aspects of deformation in shear zones it is often convenient to describe the deformation in terms of ideal simple shear. This description is justified in many cases, because there are many natural shear zones which presumably developed between rigid-wall rock blocks with zero or near-zero volume change. Under these circumstances the shear zone deformation has to be close to simple shear.

The concept of ideal simple shear has the advantage that relationships between finite strain and simple shear geometry are relatively easy. Nevertheless, when asked informally, most structural geologists will probably admit that teaching the basic aspects of shear zone deformation usually requires some concise inspection of classic handbooks before lecture hour starts. This short note aims to focus on these basic aspects using off-axis Mohr circles for stretch. Symmetric and off-axis Mohr circles of this type have been introduced to geology by Choi & Hsü (1971) and Robin (1977), respectively, and by several other workers (e.g. Means 1982, 1983, De Paor 1983, Passchier 1986). Below I use off-axis Mohr circle constructions for stretch and reciprocal stretch to present alternative proof, for the case of ideal simple shear, of some simple relationships between shear strain, the orientation and magnitude of the finite strain, and the initial orientation and rotation of a material line eventually parallel to the long axis of the strain ellipse. A number of these relationships have been documented in the literature (Ramsay 1967, Ramsay & Huber 1983). In addition to demonstrating these familiar relationships, inspection of the Mohr circle for stretch reveals some surprisingly simple and useful relationships that have been entirely overlooked in existing analyses.

Besides the likelihood that principles of a Mohr circle construction are more easily memorized than formulas, a field geologist working on natural shear zones may take profit from this approach for the purpose of rough calculations on the outcrop, just by sketching the pertinent Mohr circle constructions (Means, personal communication).

MOHR CIRCLES FOR STRETCH AND RECIPROCAL STRETCH

A general finite deformation, whether it accumulated coaxially or non-coaxially, is conveniently described by a forward (Lagrangian) position gradients tensor (F) relating the position (x) of any material point in the deformed state to coordinates (X) in the undeformed state by:

$$x_i = \mathrm{F}_{ij}X_j.$$

Conversely, the same deformation can be described by a backward (Eulerian) position gradients tensor (H) relating material coordinates (X) to spatial coordinates (x) in the deformed state:

$$X_i = \mathrm{H}_{ij}x_j$$

where the tensor H_{ij} is the inverse of F_{ij}.

Consider an ideal dextral simple shear in real space as illustrated in Fig. 1(a), with the clockwise shear strain $\gamma = \tan \psi$ taken positive. Let θ be the initial orientation of a material line, eventually oriented parallel to the long principal axis (X) of the strain ellipse at an angle θ' to the flow plane, and $\omega = \theta - \theta'$ the rotation of that material line. Such an ideal simple shear with the flow plane parallel to the one-direction can be described by:

$$\mathrm{F}_{ij} = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \quad (1a)$$

or by:

$$\mathrm{H}_{ij} = \begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix}. \quad (1b)$$

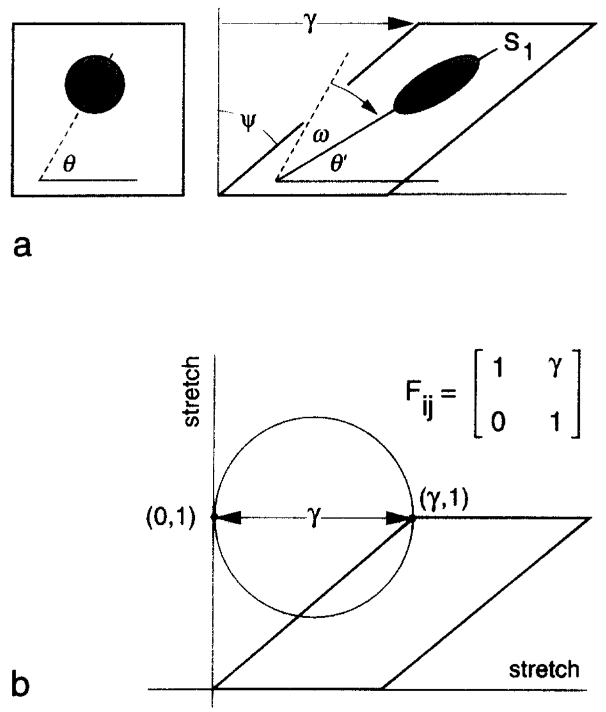

Fig. 1. (a) Geometry of ideal simple shear, with initial and final orientations of material lines parallel to the longest principal axis of the strain ellipse. Note that initial and final orientations of a material line parallel to X are symmetric about the 45° direction. For further explanation see text. (b) Mohr circle of the Second Kind representing the simple shear deformation in (a). This circle, defined by points (0, 1) and (γ, 1) spanning the diameter, overlaps with the deformation in geographic space.

Means (1983) has shown that for any tensor A_{ij} with components

$$A_{ij} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

points (a, $-c$) and (d, b) in Mohr space specify a diameter of a corresponding Mohr circle such that the Mohr circle is fully determined. Such a Mohr circle is currently known as a Mohr circle of the First Kind, as opposed to Mohr circles of the Second Kind (De Paor & Means 1984), defined in a similar fashion by points (c, a) and (b, d). The two kinds of Mohr circles differ in their sign conventions, and both kinds may have advantages and disadvantages in specific applications. A marked property of Mohr circles of the Second Kind is that they can be brought into coincidence with geographic space as shown in Fig. 1(b). As this feature seems particularly convenient for present purposes, all constructions below employ Mohr circles of the Second Kind.

Mohr circle representations of the position gradients tensors F_{ij} and H_{ij} in equations (1a & b) are shown in Fig. 2. Note that polar coordinates of points on the F Mohr circle (Fig. 2, right half) represent the stretches and rotations of material lines in real space (e.g. Means 1982). In addition, initial angles between two material lines appear as double angles measured along the circle. Material lines parallel to the finite strain axes X and Z plot at $1 + e_x$ and $1 + e_z$ along a line from the origin through the centre of the Mohr circle. The double angle 2θ represents twice the initial angle, with respect to the shear plane, of a material line eventually parallel to X. In the Mohr circle for H (Fig. 2, left half), polar coordinates of points on the circle represent the reciprocal stretches and rotations of material lines in real space, whilst angles between two material lines plotted as double angles in Mohr space now refer to angles in the deformed state. Note that the two Mohr circles are each others mirror image across the vertical axis, but that the significance of the various points indicated on the two Mohr circles are different. In particular, points representing the long and short axes of the strain ellipse swap when comparing the one with the other circle whilst, in the Mohr circle for F, point B representing a material line initially perpendicular to the flow plane, plots at B′ in the Mohr circle for H.

SIMPLE SHEAR AND FINITE STRAIN

Below we first consider mutual relationships between angles θ, θ' and ω. From the mirror symmetry of the Mohr circles for F and H it is immediately obvious that the angle OMA in the Mohr circle for F should equal $2\theta'$. As $2\theta + 2\theta' = 180°$, this implies that

$$\theta + \theta' = 90° \tag{2}$$

for any value of the shear strain. This is a rather surprising result, though consistent with analyses of simple shear such as, e.g. in Ramsay & Huber (1983). Careful examination of their fig. 2.10 on page 23 shows that the two angles should indeed be complementary. A notable consequence of equation (2) is that the initial and final orientations of material lines parallel to the longest principal axis of the strain ellipse (Fig. 1a) are symmetrically disposed about the 45° direction.

A second relationship follows from inspection of rectangular triangle OMA in the Mohr circle for F which demonstrates that $\omega + 2\theta' = 90°$ or

$$2\theta' = 90° - \omega \quad \text{or} \quad \omega = 90° - 2\theta'. \tag{3a}$$

With $\theta + \theta' = 90°$ it follows also that

$$2\theta = 90° + \omega \quad \text{or} \quad \omega = 2\theta - 90°. \tag{3b}$$

For practical purposes it may be convenient to illustrate such relationships in the Mohr circle for F only, without using the Mohr circle for H. This can be done by drawing auxiliary lines ZP and ZA as shown in Fig. 3. As arc AX equals 2θ, angle MZA spanning the same arc should be θ, and because angle MZP equals ω, angle AZP must be equal to θ'. From triangle ZAP it follows that angle ZAP should be $90°-\theta'$ such that, in the isosceles triangle ZMA, $\theta = 90°-\theta'$, and angle ZMA must be equal to $2\theta'$.

We now consider relationships between angles θ, θ' and ω, and the value of the shear strain γ. By definition, $\gamma = \tan\psi$ in which ψ is the rotation of a material line initially perpendicular to the flow plane. This material

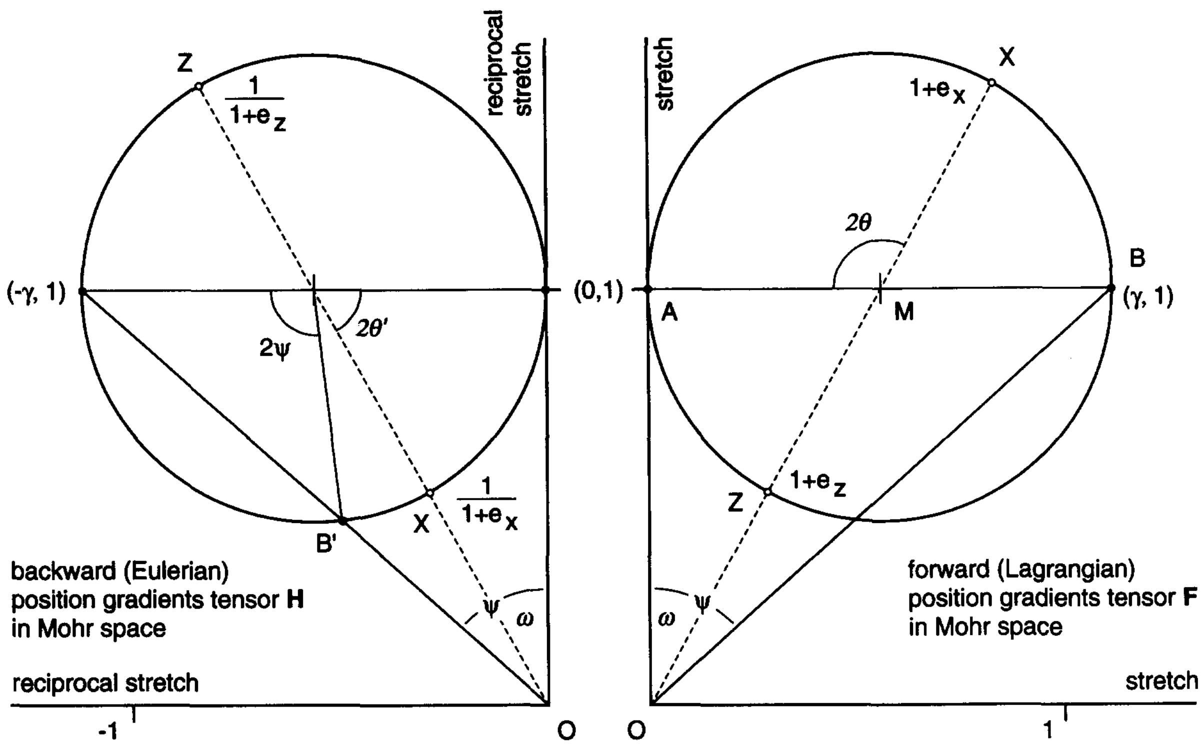

Fig. 2. Mohr circle representations for F (right-hand half) and H (left-hand half) for an ideal simple shear as shown in Fig. 1.

line is represented in the Mohr circle for F by point B (Fig. 2). Note that this point B is the pole on the Mohr circle (Allison 1984). With OA being unit length, the diameter BA of the circle must be equal to γ, which is also inherent to the procedure to plot Mohr circles from tensor components as laid out by Means (1983). It

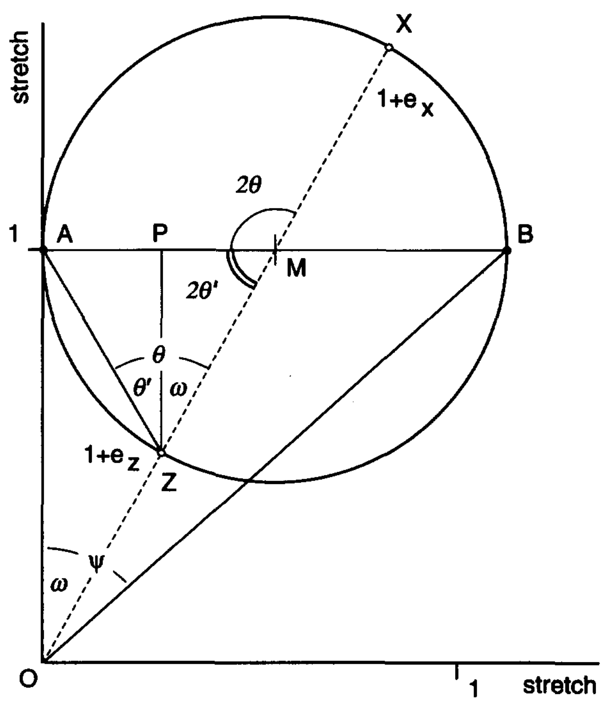

Fig. 3. Mohr circle representation for F, with auxiliary lines ZP and ZA to visualize the orientation θ' of the longest principal axis of strain.

follows that the radius of the circle should be $\gamma/2$. Therefore, from rectangular triangle OMA:

$$\tan\omega = \gamma/2. \tag{4}$$

This is obviously consistent with the somewhat complicated calculations on the basis of the geometry of simple shear in real space as, e.g. in Ramsay & Huber (1983, pp. 17 and 27). From the same triangle, it also follows that

$$\tan 2\theta' = 2/\gamma \tag{5a}$$

which only differs in sign from Ramsay & Huber (1983, p. 27) due to the present sign convention with clockwise shear strains positive. With $2\theta + 2\theta' = 180°$ and bearing in mind from trigonometry that $\tan(180° - \alpha) = -\tan\alpha$, it follows that

$$\tan 2\theta = -2/\gamma. \tag{5b}$$

We have noted that the diameters of the Mohr circles for F and H equal the shear strain γ. This diameter is also equal to the difference between the largest and smallest principal stretches $1 + e_x$ and $1 + e_z$, such that

$$\gamma = (1 + e_x) - (1 + e_z) = e_x - e_z \tag{6}$$

and, with (4)

$$\tan\omega = (e_x - e_z)/2. \tag{7}$$

The result under (6), though surprising, is entirely consistent with conventional analyses of simple shear as for example laid out in Ramsay (1967), where close inspection of his fig. 3.21 (p. 85) indeed indicates that the difference between the principal axes of strain is equal to the shear strain. To my knowledge, the only recent reference to this property of simple shear has been made by Treagus (1981) where she draws attention to an extremely elegant simple shear construction set out by Thomson & Tait (1867). With reference to Durelli *et al.* (1958), De Paor (1983) has identified the circle of the Thomson & Tait (1867) construction as a dyadic circle concentric with the Mohr circle for F. It will be clear that some of the geometrical properties shown above also emerge from this dyadic circle.

Now to calculate the principal axes (stretches), inspect rectangular triangle OMA. Note that OM = $\frac{1}{2}[(1 + e_x) + (1 + e_z)]$ and AM = $\gamma/2$. With Pythagoras' rule and multiplying by 4 it follows that

$$((1 + e_x) + (1 + e_z))^2 = 4 + \gamma^2. \tag{8}$$

With (6) this equation is easily solved for either $(1 + e_x)$ or $(1 + e_z)$ to produce the well known expressions for the principal axes as a function of shear strain (Ramsay 1967, Ramsay & Huber 1983):

$$(1 + e_x)^2 \quad \text{or} \quad (1 + e_z)^2 = \tfrac{1}{2}[\gamma^2 + 2 \pm \gamma(\gamma^2 + 4)^{1/2}].$$

CONCLUSION

The Mohr circles for stretch and reciprocal stretch allow convenient proofs of a number of simple relationships between orientation and magnitude of the strain built up in simple shear, and the initial orientation and rotation of material lines that become parallel to the longest principal axis. Inspection of the Mohr circles for stretch also allows one to assess some relationships commonly not documented in algebraic representations of simple shear.

Acknowledgements—The idea to derive the above relationships with the Mohr circle for stretch was born during a lively discussion with John Platt on the complexities of simple shear. I thank Sue Treagus for drawing my attention to her paper on the Thomson and Tait construction, and for sending me photocopies of extracts from the original Thomson and Tait publication. Win Means is kindly thanked for his accurate comments and for copies of his Short Course on the use of Mohr circles. I feel much obliged to Declan De Paor whose detailed review helped to considerably clarify my understanding of different Mohr circle conventions.

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